RIEMANNIAN MANIFOLDS WITH POSITIVE YAMABE INVARIANT AND PANEITZ OPERATOR

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Abstract. For a Riemannian manifold with dimension at least six, we prove that the existence of a conformal metric with positive scalar and Q curvature is equivalent to the positivity of both the Yamabe invariant and the Paneitz operator.

1. Introduction

Let (M,g) be a smooth compact Riemannian manifold with dimension $n \geq 3$. Denote $[g]$ as the conformal class of metrics associated with g. The Yamabe invariant is given by (see [LP])

$$Y(M,g) = \inf_{\widetilde{g}\in[g]} \frac{\int_M \widetilde{R} d\widetilde{\mu}}{\widetilde{\mu}(M)^{\frac{n-2}{n}}}, \tag{1.1}$$

here $\widetilde{R}$ is the scalar curvature of $\widetilde{g}$ and $\widetilde{\mu}$ is the measure associated with $\widetilde{g}$. In terms of the conformal Laplacian operator

$$L = -\frac{4(n-1)}{n-2}\Delta + R, \tag{1.2}$$

we have

$$\begin{aligned} Y(M,g) &= \inf_{\substack{u\in C^\infty(M)\\ u>0}} \frac{\int_M Lu\cdot u d\mu}{\|u\|^2_{L^{\frac{2n}{n-2}}}} \qquad (1.3)\\ &= \inf_{u\in H^1(M)\backslash\{0\}} \frac{\int_M \left(\frac{4(n-1)}{n-2}|\nabla u|^2 + Ru^2\right) d\mu}{\|u\|^2_{L^{\frac{2n}{n-2}}}}. \end{aligned}$$

In particular $Y(M,g)>0$ if and only if the first eigenvalue $\lambda_1(L)>0$. Moreover based on the fact that the first eigenfunction of L must be strictly positive or negative, we know

$$\lambda_1(L) > 0 \Leftrightarrow \text{there exists a } \widetilde{g}\in[g] \text{ with } \widetilde{R}>0. \tag{1.4}$$

Here we are looking for similar characterization for Paneitz operator and Q curvature (see [B, P]). More precisely we are interested in the solution to

Problem 1.1. *For a Riemannian manifold with dimension at least five, can we find a conformal invariant condition which is equivalent to the existence of a conformal metric with positive scalar and Q curvature?*

In view of the results on positivity of Paneitz operator in [GM, XY], many people suspect the conformal invariant condition wanted in Problem 1.1 should be the positivity of Yamabe invariant and Paneitz operator. As a consequence of the

main result below, this is verified for dimension at least six (see Corollary 1.1). It is very likely the statement is still true for dimension five. However due to technical constrains in our approach, dimension five case remains an open problem.

To write down the formula of Q curvature and Paneitz operator, following [B], let

$$J=\frac{R}{2\left(n-1\right)},\quad A=\frac{1}{n-2}\left(Rc-Jg\right), \tag{1.5}$$

here Rc is the Ricci curvature tensor. The Q curvature is given by

$$Q=-\Delta J-2\left|A\right|^{2}+\frac{n}{2}J^{2}. \tag{1.6}$$

The Paneitz operator is given by

$$P\varphi=\Delta^{2}\varphi+\operatorname{div}\left(4A\left(\nabla\varphi,e_{i}\right)e_{i}-\left(n-2\right)J\nabla\varphi\right)+\frac{n-4}{2}Q\varphi. \tag{1.7}$$

Here $e_1,\cdots,e_n$ is a local orthonormal frame with respect to g. For $n\geq 5$, under a conformal change of metric, the Paneitz operator satisfies

$$P_{\rho^{\frac{4}{n-4}}g}\varphi=\rho^{-\frac{n+4}{n-4}}P_{g}\left(\rho\varphi\right), \tag{1.8}$$

(see [B]); compare to the conformal covariant property of the conformal Laplacian (1.2). In addition, the Q curvature is transformed by the formula

$$Q_{\rho^{\frac{4}{n-4}}g}=\frac{2}{n-4}\rho^{-\frac{n+4}{n-4}}P_{g}\rho. \tag{1.9}$$

We now define two conformal invariants related to the Q-curvature. First, in analogy with the Yamabe invariant, we define

$$Y_{4}^{+}\left(M,g\right)=\frac{n-4}{2}\inf_{\widetilde{g}\in[g]}\frac{\int_{M}\widetilde{Q}d\widetilde{\mu}}{\left(\widetilde{\mu}\left(M\right)\right)^{\frac{n-4}{n}}}=\inf_{\substack{u\in C^{\infty}(M)\\ u>0}}\frac{\int_{M}Pu\cdot ud\mu}{\left\|u\right\|_{L^{\frac{2n}{n-4}}}^{2}}. \tag{1.10}$$

We use $Y_4^+(M,g)$ to emphasize the infimum is taken over positive functions (i.e. conformal factors). To define the second invariant, denote

$$\begin{aligned} E\left(\varphi\right) &= \int_{M}P\varphi\cdot\varphi d\mu \qquad (1.11)\\ &= \int_{M}\left(\left(\Delta\varphi\right)^{2}-4A\left(\nabla\varphi,\nabla\varphi\right)+\left(n-2\right)J\left|\nabla\varphi\right|^{2}+\frac{n-4}{2}Q\varphi^{2}\right)d\mu. \end{aligned}$$

It is clear that $E(\varphi)$ is well defined for $\varphi\in H^2(M)$. Define

$$Y_{4}\left(M,g\right)=\inf_{u\in H^{2}(M)\backslash\{0\}}\frac{E\left(u\right)}{\left\|u\right\|_{L^{\frac{2n}{n-4}}}^{2}}. \tag{1.12}$$

Clearly

$$Y_{4}\left(M,g\right)\leq Y_{4}^{+}\left(M,g\right),$$

but in general (and in contrast to the usual Yamabe invariant) we have an inequality instead of equality, due to the fact Paneitz operator is fourth order. Note that from standard elliptic theory $Y_4(M,g)>0$ if and only if the first eigenvalue $\lambda_1(P)>0$ i.e. P is positive definite.

When the Yamabe invariant $Y(M,g)>0$, there is another closely related quantity $Y_4^*(M,g)$ defined by

$$Y_4^*(M,g)=\frac{n-4}{2}\inf_{\substack{\widetilde{g}\in[g]\\ \widetilde{R}>0}}\frac{\int_M \widetilde{Q}d\widetilde{\mu}}{\left(\widetilde{\mu}(M)\right)^{\frac{n-4}{n}}}. \tag{1.13}$$

Obviously

$$Y_4(M,g)\le Y_4^+(M,g)\le Y_4^*(M,g). \tag{1.14}$$

One of the main goals of this paper is to understand the relationship between these quantities, and their connection to the existence of a metric with positive scalar and Q-curvature.

Our motivation for studying these problems comes from recent progress in the understanding of Q curvature equations for dimension at least five in [GM, HY1, HY2]. Paneitz operator and Q curvature were brought to attention in [CGY]. For dimension at least five, in various work [HeR, HuR, R], people had realized the important role of positivity of Green's function of Paneitz operator in understanding the Q curvature equations. Such kind of positivity is hard to get due to the lack of maximum principle for fourth order equations. A breakthrough was achieved in [GM], namely for $n\ge 5$, if $R>0$ and $Q>0$, then $P>0$ and the Green's function of Paneitz operator G_P is strictly positive. Subsequently in [HY1], for positive Yamabe invariant case it was found the positivity of Green's function of Paneitz operator is equivalent to the existence of a conformal metric with positive Q curvature. Indeed it was shown that if $n\ge 5$, $Y(M,g)>0$, then there exists a conformal metric with positive Q curvature if and only if $\ker P=0$ and $G_P>0$, it is also equivalent to $\ker P=0$ and $G_{P,p}>0$ for a fixed p. Note the positivity of Green's function is a conformal invariant condition. In [GM], it was shown that $R>0$ and $Q>0$ implies $Y_4(M,g)$ is achieved at a positive smooth function. The assumption was relaxed to $Y(M,g)>0, Q>0$ and $P>0$ in [HY2]. Trying to understand relations between various assumptions motivates problems considered here.

Theorem 1.1. *Let (M,g) be a smooth compact Riemannian manifold with dimension $n\ge 6$. If $Y(M,g)>0$ and $Y_4^*(M,g)>0$, then there exists a metric $\widetilde{g}\in[g]$ satisfying $\widetilde{R}>0$ and $\widetilde{Q}>0$.*

Combine Theorem 1.1 with existence and positivity results in [GM, HY2, XY] we have the following corollaries.

Corollary 1.1. *Let (M,g) be a smooth compact Riemannian manifold with dimension $n\ge 6$. Then the following statements are equivalent*

(1) $Y(M,g)>0, P>0$.
(2) $Y(M,g)>0, Y_4^*(M,g)>0$.
(3) *there exists a metric $\widetilde{g}\in[g]$ satisfying $\widetilde{R}>0$ and $\widetilde{Q}>0$.*

Corollary 1.1 answers Problem 1.1 for dimension at least six.

Corollary 1.2. *Let (M,g) be a smooth compact Riemannian manifold with dimension $n\ge 6$. If $Y(M,g)>0$ and $Y_4^*(M,g)>0$, then $P>0$, the Green's function $G_P>0$, and $Y_4(M,g)$ is achieved at a positive smooth function u with $R_{u^{\frac{4}{n-4}}g}>0$ and $Q_{u^{\frac{4}{n-4}}g}=const$. In particular,*

$$Y_4(M,g)=Y_4^+(M,g)=Y_4^*(M,g).$$

The dimensional restriction $n \geq 6$ is an unfortunate by-product of our technique and it is very likely the result holds in dimension five as well. To explain our approach, we first point out that the Q curvature equation is variational: a metric has constant Q curvature if and only if it is a critical point of the total Q curvature functional $\int_M Q_g d\mu_g$ with g running through the set of conformal metrics with unit volume. A closely related quantity is

$$\sigma_2(A) = \frac{1}{2}\left(J^2 - |A|^2\right). \tag{1.15}$$

Indeed $\sigma_2(A)$ equation is also variational in similar sense. Since

$$Q = -\Delta J + \frac{n-4}{2}J^2 + 4\sigma_2(A), \tag{1.16}$$

we have

$$\int_M Q d\mu = \frac{n-4}{2}\int_M J^2 d\mu + 4\int_M \sigma_2(A)\, d\mu. \tag{1.17}$$

Obviously $\int_M J^2 d\mu$ is always nonnegative. For $t \geq 1$ consider the functional

$$\frac{n-4}{2}t\int_M J_g^2 d\mu_g + 4\int_M \sigma_2(A_g)\, d\mu_g. \tag{1.18}$$

A critical metric of this functional restricted to the space of conformal metrics of unit volume satisfies

$$t\left(-\Delta J + \frac{n-4}{2}J^2\right) + 4\sigma_2(A) = const. \tag{1.19}$$

In the appendix, we will use elementary methods to show that if $Y(M,g) > 0$, then there exists $g_0 \in [g]$, $g_0 = u_0^{\frac{4}{n-4}}g$ and $t_0 \gg 1$ such that

$$t_0\left(-\Delta_0 J_0 + \frac{n-4}{2}J_0^2\right) + 4\sigma_2(A_0) > 0 \tag{1.20}$$

and $J_0 > 0$. Define f as

$$t_0\left(-\Delta_0 J_0 + \frac{n-4}{2}J_0^2\right) + 4\sigma_2(A_0) = f u_0^{-\frac{n+4}{n-4}}. \tag{1.21}$$

Then for $1 \leq t \leq t_0$ we consider the following 1-parameter family of equations:

$$t\left(-\widetilde{\Delta} J + \frac{n-4}{2}\widetilde{J}^2\right) + 4\sigma_2\left(\widetilde{A}\right) = f u^{-\frac{n+4}{n-4}}, \quad \widetilde{g} = u^{\frac{4}{n-4}}g. \tag{1.22}$$

Let

$$\begin{aligned} & S \\ = & \left\{t \in [1,t_0] : \text{there exists positive smooth } u \text{ solving (1.22), with } \widetilde{R} > 0\right\}. \end{aligned} \tag{1.23}$$

Then $t_0 \in S$. We will show S is both open and closed by the implicit function theorem and apriori estimates. Hence $S = [1, t_0]$. It follows that there exists a $\widetilde{g} \in [g]$ with $\widetilde{R} > 0$ and $\widetilde{Q} > 0$.

We conclude the introduction with some remarks. The dimensional restriction $n \geq 6$ appears in both the open and closed part of the argument. The power of the conformal factor u on the right hand side of (1.22) is chosen to be negative to give better estimate of solutions. Moreover this choice also leads to a good sign of the zeroth order term in the linearized operator. We also observe that our path of equations is variational, it has a divergence structure which we will exploit in

apriori estimates. At last we note that in dimension four a path of equations which is analogous to (1.22) is considered in [CGY] to produce a conformal metric with positive scalar curvature and $\sigma_2(A)$ curvature assuming the positivity of Yamabe invariant and conformal invariant $\int_M \sigma_2(A)\, d\mu$.

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2. The method of continuity and openness

In this section we set up the continuity method. It will be more convenient if we first rewrite equation (1.22) in terms of Q and $\sigma_2(A)$. Using (1.16), equation (1.22) can be expressed as

$$t\left(\widetilde{Q}-4\sigma_2\left(\widetilde{A}\right)\right)+4\sigma_2\left(\widetilde{A}\right)=fu^{-\frac{n+4}{n-4}}, \tag{2.1}$$

hence

$$t\widetilde{Q}+4(1-t)\,\sigma_2\left(\widetilde{A}\right)=fu^{-\frac{n+4}{n-4}}. \tag{2.2}$$

Dividing by t (recall $t\geq 1$) and denoting

$$\lambda=\frac{4(t-1)}{t},\quad \chi=\frac{f}{t}, \tag{2.3}$$

equation (1.22) is equivalent to

$$\widetilde{Q}-\lambda\sigma_2\left(\widetilde{A}\right)=\chi u^{-\frac{n+4}{n-4}}, \tag{2.4}$$

here $\widetilde{g}=u^{\frac{4}{n-4}}g$ and $0\leq\lambda<4$.

To write this equation in terms of the conformal factor u, observe that the Schouten tensor of the conformal metric $\widetilde{g}$ is given by

$$\widetilde{A}_{ij}=A_{ij}-\frac{2}{n-4}u^{-1}u_{ij}-\frac{2}{(n-4)^2}u^{-2}\left|\nabla u\right|^2 g_{ij}+\frac{2(n-2)}{(n-4)^2}u^{-2}u_iu_j, \tag{2.5}$$

hence

$$\begin{aligned}&\sigma_2\left(\widetilde{A}\right) \qquad\qquad (2.6)\\ =\;& u^{-\frac{8}{n-4}}\left[\sigma_2(A)+\frac{2}{(n-4)^2}u^{-2}(\Delta u)^2-\frac{2}{(n-4)^2}u^{-2}\left|D^2u\right|^2+\frac{4}{(n-4)^3}u^{-3}\left|\nabla u\right|^2\Delta u\right.\\ &+\frac{4(n-2)}{(n-4)^3}u^{-3}u_{ij}u_iu_j-\frac{2}{n-4}Ju^{-1}\Delta u+\frac{2}{n-4}u^{-1}A_{ij}u_{ij}-\frac{2(n-1)}{(n-4)^3}u^{-4}\left|\nabla u\right|^4\\ &\left.-\frac{2}{(n-4)^2}Ju^{-2}\left|\nabla u\right|^2-\frac{2(n-2)}{(n-4)^2}u^{-2}A_{ij}u_iu_j\right].\end{aligned}$$

Using the formula (1.9) we also have

$$\widetilde{Q}=\frac{2}{n-4}u^{-\frac{n+4}{n-4}}Pu \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.4), then multiplying through by $\frac{n-4}{2}u^{\frac{n+4}{n-4}}$ we have

$$\begin{aligned} & Pu \qquad (2.8) \\ = \; & \lambda u \left[\frac{n-4}{2} \sigma_2 (A) + \frac{1}{n-4} u^{-2} (\Delta u)^2 - \frac{1}{n-4} u^{-2} \left| D^2 u \right|^2 + \frac{2}{(n-4)^2} u^{-3} |\nabla u|^2 \Delta u \right. \\ & + \frac{2(n-2)}{(n-4)^2} u^{-3} u_{ij} u_i u_j - J u^{-1} \Delta u + u^{-1} A_{ij} u_{ij} - \frac{n-1}{(n-4)^2} u^{-4} |\nabla u|^4 \\ & \left. - \frac{1}{n-4} J u^{-2} |\nabla u|^2 - \frac{n-2}{n-4} u^{-2} A_{ij} u_i u_j \right] + \frac{n-4}{2} \chi . \end{aligned}$$

Also, the condition that $\widetilde{J} > 0$ is equivalent to the inequality

$$\Delta u < -\frac{2}{n-4} u^{-1} |\nabla u|^2 + \frac{n-4}{2} J u. \tag{2.9}$$

We begin with a fact which permits us to start the continuity process.

Proposition 2.1. *Let (M, g) be a smooth compact Riemannian manifold with dimension $n \geq 5$. If $Y(M, g) > 0$, then there exists a metric $\widetilde{g} \in [g]$ with*

$$-\widetilde{\Delta} \widetilde{J} + \frac{n-4}{2} \widetilde{J}^2 > 0, \quad \widetilde{J} > 0. \tag{2.10}$$

Note this proposition is clearly a consequence of the solution to the Yamabe problem ([LP]). In the Appendix we will provide an elementary proof. In view of Proposition 2.1 we can find $\widetilde{g} = u_0^{\frac{4}{n-4}} g$ such that

$$-\widetilde{\Delta} \widetilde{J} + \frac{n-4}{2} \widetilde{J}^2 > 0, \quad \widetilde{J} > 0. \tag{2.11}$$

Since

$$\widetilde{Q} - \lambda \sigma_2 \left(\widetilde{A} \right) = -\widetilde{\Delta} \widetilde{J} + \frac{n-4}{2} \widetilde{J}^2 + (4 - \lambda) \sigma_2 \left(\widetilde{A} \right), \tag{2.12}$$

we can find $0 < \lambda_0 < 4$ close enough to 4 such that $\widetilde{Q} - \lambda_0 \sigma_2 \left(\widetilde{A} \right) > 0$. In particular $\widetilde{g} = u_0^{\frac{4}{n-4}} g$ is a solution of (2.4), with

$$\chi = \left(\widetilde{Q} - \lambda_0 \sigma_2 \left(\widetilde{A} \right) \right) u_0^{\frac{n+4}{n-4}} \tag{2.13}$$

Define

$$\Sigma = \left\{ 0 \leq \lambda \leq \lambda_0 : \exists u \in C^\infty (M), u > 0, \text{ satisfying (2.4) and } \widetilde{J} > 0 \right\}. \tag{2.14}$$

It is clear that $\lambda_0 \in \Sigma$. Indeed in this case u_0 is a solution. On the other hand if we can show $0 \in \Sigma$, then it follows there exists a metric $\widetilde{g} \in [g]$ with $\widetilde{J} > 0$ and $\widetilde{Q} > 0$. To achieve this we will show Σ is both open and closed.

To prove that Σ is open, we consider the linearized operator. To this end, define the map

$$\widetilde{g} = u^{\frac{4}{n-4}} g \mapsto \mathcal{N}[u] = \widetilde{Q} - \lambda \sigma_2 \left(\widetilde{A} \right) - \chi u^{-\frac{n+4}{n-4}}. \tag{2.15}$$

Then $\mathcal{N}[u] = 0$ if and only if $\widetilde{g}$ is a solution of (2.4). Let $\widetilde{S}$ denote the linearization of $\mathcal{N}$ at u:

$$\widetilde{S} \varphi = \left. \frac{d}{dt} \right|_{t=0} \mathcal{N}[u + t\varphi]. \tag{2.16}$$

To compute $\widetilde{S}$ we use the standard formulas for the variation of the Q curvature and the Schouten tensor

$$\left.\frac{d}{dt}\right|_{t=0} Q_{(1+t\psi)g} = \frac{1}{2}P_g\psi - \frac{n+4}{4}Q_g\psi, \tag{2.17}$$

$$\left.\frac{d}{dt}\right|_{t=0} \sigma_2\left(A_{(1+t\psi)g}\right) = -\frac{1}{2}J_g\Delta_g\psi + \frac{1}{2}g\left(A_g, D_g^2\psi\right) - 2\sigma_2\left(A_g\right)\psi. \tag{2.18}$$

In our setting, using

$$(u+t\varphi)^{\frac{4}{n-4}} g = \left(1+tu^{-1}\varphi\right)^{\frac{4}{n-4}} \widetilde{g} \tag{2.19}$$

and

$$\left.\frac{d}{dt}\right|_{t=0} \left(1+tu^{-1}\varphi\right)^{\frac{4}{n-4}} = \frac{4}{n-4}u^{-1}\varphi, \tag{2.20}$$

we see

$$\widetilde{S}\varphi = \frac{2}{n-4}\widetilde{H}\left(u^{-1}\varphi\right), \tag{2.21}$$

where

$$\begin{aligned} &\widetilde{H}\varphi \\ = \; &\widetilde{P}\varphi - \frac{n+4}{2}\widetilde{Q}\varphi + \lambda\left(\widetilde{J}\widetilde{\Delta}\varphi - \widetilde{g}\left(\widetilde{A}, \widetilde{D}^2\varphi\right) + 4\sigma_2\left(\widetilde{A}\right)\varphi\right) + \frac{n+4}{2}\chi u^{-\frac{n+4}{n-4}}\varphi. \end{aligned} \tag{2.22}$$

The openness of Σ follows from implicit function theorem and standard elliptic theory, together with the following lemma:

Lemma 2.1. *Let (M, g) be a smooth compact Riemannian manifold with dimension $n \geq 6$. If $0 \leq \lambda \leq 4$,*

$$Q - \lambda\sigma_2(A) > 0, \quad J > 0, \tag{2.23}$$

then the operator

$$\begin{aligned} &H\varphi \\ = \; &P\varphi - \frac{n+4}{2}Q\varphi + \lambda\left(J\Delta\varphi - A_{ij}\varphi_{ij} + 4\sigma_2(A)\varphi\right) + \frac{n+4}{2}\left(Q - \lambda\sigma_2(A)\right)\varphi \end{aligned} \tag{2.24}$$

is positive definite.

Proof. For any smooth function φ,

$$\begin{aligned} &\int_M H\varphi\cdot\varphi d\mu \\ = &\int_M \left[(\Delta\varphi)^2 + (n-2-\lambda)J|\nabla\varphi|^2 - (4-\lambda)A(\nabla\varphi,\nabla\varphi) + \frac{n-4}{2}\left(Q-\lambda\sigma_2(A)\right)\varphi^2\right]d\mu. \end{aligned} \tag{2.25}$$

Using the Bochner formula

$$\int_M (\Delta\varphi)^2 d\mu = \int_M \left|D^2\varphi\right|^2 d\mu + \int_M J|\nabla\varphi|^2 d\mu + (n-2)\int_M A(\nabla\varphi,\nabla\varphi)d\mu, \tag{2.26}$$

we see

$$\begin{aligned} &\int_M H\varphi\cdot\varphi d\mu \\ = &\frac{n-6+\lambda}{n-2}\int_M (\Delta\varphi)^2 d\mu + \frac{4-\lambda}{n-2}\int_M \left|D^2\varphi\right|^2 d\mu \\ &+\frac{4-\lambda+(n-2)(n-2-\lambda)}{n-2}\int_M J|\nabla\varphi|^2 d\mu + \frac{n-4}{2}\int_M \left(Q-\lambda\sigma_2(A)\right)\varphi^2 d\mu. \end{aligned} \tag{2.27}$$

Note for $n \geq 6$ and $0 \leq \lambda \leq 4$, all coefficients before the integral sign are nonnegative. As a consequence H is positive definite. ∎

For $n = 5$, note that the coefficient of $\int_M J |\nabla \varphi|^2 d\mu$ is equal to $\frac{13-4\lambda}{3}$ and it is negative when λ is close to 4.

3. Apriori estimate

In this section we prove apriori estimate for smooth positive solutions to (2.4) with positive scalar curvature for $0 \leq \lambda \leq \lambda_0$. An immediate consequence is that the set Σ (see (2.14)) is closed.

Lemma 3.1. *Assume (M, g) is a smooth compact Riemannian manifold with dimension $n \geq 6$. If $Y(M, g) > 0$, $Y_4^*(M, g) > 0$ and $0 \leq \lambda \leq \lambda_0 < 4$, then any smooth positive solution u to (2.4) with $\widetilde{J} > 0$ satisfies*

$$\|u\|_{L^{\frac{2n}{n-4}}} \leq c \tag{3.1}$$

and

$$u \geq c > 0. \tag{3.2}$$

Here c is independent of u and λ.

Proof. Let $\widetilde{g} = u^{\frac{4}{n-4}} g$. Using

$$\int_M \widetilde{Q} d\widetilde{\mu} = 4 \int_M \sigma_2\left(\widetilde{A}\right) d\widetilde{\mu} + \frac{n-4}{2} \int_M \widetilde{J}^2 d\widetilde{\mu}, \tag{3.3}$$

and (2.4) we get

$$\left(1 - \frac{\lambda}{4}\right) \int_M \widetilde{Q} d\widetilde{\mu} + \frac{n-4}{8} \lambda \int_M \widetilde{J}^2 d\widetilde{\mu} = \int_M \chi u^{-\frac{n+4}{n-4}} d\widetilde{\mu} = \int_M \chi u d\mu. \tag{3.4}$$

By the definition of $Y_4^*(M, g)$ (see (1.13)) we have

$$\begin{aligned} \|u\|^2_{L^{\frac{2n}{n-4}}} &= \widetilde{\mu}(M)^{\frac{n-4}{n}} \\ &\leq \frac{n-4}{2} \frac{1}{Y_4^*(M, g)} \int_M \widetilde{Q} d\widetilde{\mu} \\ &\leq c \int_M \chi u d\mu \\ &\leq c \|u\|_{L^{\frac{2n}{n-4}}}. \end{aligned} \tag{3.5}$$

Hence

$$\|u\|_{L^{\frac{2n}{n-4}}} \leq c. \tag{3.6}$$

Multiplying both sides of equation (2.8) by u^α and doing integration by parts we get

$$\begin{aligned} &\frac{n-4}{2} \int_M \chi u^\alpha d\mu \\ = \ & \alpha \int_M u^{\alpha-1} (\Delta u)^2 d\mu + \left[\alpha(\alpha-1) + \frac{3\alpha-1}{2(n-4)} \lambda\right] \int_M u^{\alpha-2} |\nabla u|^2 \Delta u d\mu \\ &+ \frac{(n-4)\alpha^2 - (n-8)\alpha - 2}{2(n-4)^2} \lambda \int_M u^{\alpha-3} |\nabla u|^4 d\mu + (n-2-\lambda)\alpha \int_M J u^{\alpha-1} |\nabla u|^2 d\mu \\ &- (4-\lambda)\alpha \int_M u^{\alpha-1} A(\nabla u, \nabla u) d\mu + \frac{n-4}{2} \int_M (Q - \lambda \sigma_2(A)) u^{\alpha+1} d\mu. \end{aligned} \tag{3.7}$$

If
$$\alpha(\alpha-1)+\frac{3\alpha-1}{2(n-4)}\lambda>0, \tag{3.8}$$
(this happens when $|\alpha|$ is large enough), then using (2.9) we get
$$\begin{aligned}&\frac{n-4}{2}\int_M \chi u^{\alpha}d\mu \\ \leq\ & \alpha\int_M u^{\alpha-1}(\Delta u)^2 d\mu-\left[\frac{4-\lambda}{2(n-4)}\alpha^2+\frac{16-2\lambda-n(4-\lambda)}{2(n-4)^2}\alpha\right]\int_M u^{\alpha-3}|\nabla u|^4 d\mu \\ &+\left(\frac{n-4}{2}\alpha^2+\frac{2n-\lambda}{4}\alpha-\frac{\lambda}{4}\right)\int_M Ju^{\alpha-1}|\nabla u|^2 d\mu-(4-\lambda)\alpha\int_M u^{\alpha-1}A(\nabla u,\nabla u)\,d\mu \\ &+\frac{n-4}{2}\int_M (Q-\lambda\sigma_2(A))\,u^{\alpha+1}d\mu.\end{aligned} \tag{3.9}$$
Now let $\alpha=-(p+1)$ with $p\gg 1$, using $\lambda\leq\lambda_0<4$ and the fact for all $\varepsilon>0$,
$$u^{-p-2}|\nabla u|^2\leq \varepsilon u^{-p-4}|\nabla u|^4+\frac{1}{4\varepsilon}u^{-p},$$
we get
$$\begin{aligned}\int_M u^{-p-1}d\mu\ \ \leq\ \ & -c\int_M u^{-p-2}(\Delta u)^2 d\mu-c\int_M u^{-p-4}|\nabla u|^4 d\mu \\ &+c\int_M u^{-p}d\mu.\end{aligned} \tag{3.10}$$
Hence
$$\left\|u^{-1}\right\|_{L^{p+1}}^{p+1}\leq c\left\|u^{-1}\right\|_{L^{p}}^{p}\leq c\left\|u^{-1}\right\|_{L^{p+1}}^{p}. \tag{3.11}$$
It follows that
$$\left\|u^{-1}\right\|_{L^{p+1}}\leq c. \tag{3.12}$$
To continue let $u^{-1}=U$, then the inequality (2.9) implies
$$\begin{aligned}-\Delta U\ \ <\ \ & -\frac{2(n-3)}{n-4}U^{-1}|\nabla U|^2+\frac{n-4}{2}JU \\ \leq\ \ & \frac{n-4}{2}JU.\end{aligned} \tag{3.13}$$
By [GT, Theorem 8.17 on p194], (3.12) and (3.13) together imply $U\leq c$, in another word
$$u\geq c>0. \tag{3.14}$$
∎

To derive further estimates on u, we denote
$$v=u^{-q-1}\left(-\Delta u+\frac{n-4}{2}Ju\right), \tag{3.15}$$
here $q\geq 0$ is a number to be determined. It follows from (2.9) that
$$v>\frac{2}{n-4}u^{-q-2}|\nabla u|^2. \tag{3.16}$$
Since
$$\Delta u=\frac{n-4}{2}Ju-u^{q+1}v, \tag{3.17}$$

we see that under local orthonormal frame with respect to g,

$$\begin{aligned}\Delta^2 u &= -u^{q+1}\Delta v - 2(q+1)u^q u_i v_i + (q+1)u^{2q+1}v^2 \\ &\quad -q(q+1)u^{q-1}|\nabla u|^2 v - \frac{n-4}{2}(q+2)Ju^{q+1}v \\ &\quad +(n-4)J_i u_i + \frac{(n-4)^2}{4}J^2 u + \frac{n-4}{2}\Delta J\cdot u.\end{aligned}\tag{3.18}$$

Plug (3.17) and (3.18) into (2.8) we get

$$\begin{aligned}&-\Delta v - 2(q+1)u^{-1}u_i v_i + (q+1)u^q v^2 - q(q+1)u^{-2}|\nabla u|^2 v \\ &+\left(-\frac{n-4}{2}q+2\right)Jv + 4u^{-q-1}A_{ij}u_{ij} + 2u^{-q-1}J_i u_i - (n-4)|A|^2 u^{-q} \\ =\ & \frac{\lambda}{n-4}u^q v^2 - \frac{2\lambda}{(n-4)^2}u^{-2}|\nabla u|^2 v - \frac{\lambda}{n-4}u^{-q-2}\left|D^2 u\right|^2 + \frac{2(n-2)\lambda}{(n-4)^2}u^{-q-3}u_{ij}u_i u_j \\ &+\lambda u^{-q-1}A_{ij}u_{ij} - \frac{(n-1)\lambda}{(n-4)^2}u^{-q-4}|\nabla u|^4 - \frac{(n-2)\lambda}{n-4}u^{-q-2}A_{ij}u_i u_j - \frac{(n-4)\lambda}{4}|A|^2 u^{-q} \\ &+\frac{n-4}{2}\chi u^{-q-1}.\end{aligned}\tag{3.19}$$

Multiplying v^α on both sides and doing integration by parts we have

$$\begin{aligned}&\alpha v^{\alpha-1}|\nabla v|^2 + \frac{(\alpha-1)(q+1)}{\alpha+1}u^q v^{\alpha+2} - (q+1)\left(q+\frac{2}{\alpha+1}\right)u^{-2}|\nabla u|^2 v^{\alpha+1} \\ &-4\alpha u^{-q-1}A_{ij}u_i v_j v^{\alpha-1} + 4(q+1)u^{-q-2}A_{ij}u_i u_j v^\alpha + \left[-\frac{(n-4)(\alpha-1)}{2(\alpha+1)}q + \frac{2\alpha+n-2}{\alpha+1}\right]Jv^{\alpha+1} \\ &-2u^{-q-1}J_i u_i v^\alpha - (n-4)|A|^2 u^{-q}v^\alpha \\ \sim\ & \frac{\lambda}{n-4}u^q v^{\alpha+2} - \frac{2\lambda}{(n-4)^2}u^{-2}|\nabla u|^2 v^{\alpha+1} - \frac{\lambda}{n-4}u^{-q-2}\left|D^2u\right|^2 v^\alpha + \frac{2(n-2)\lambda}{(n-4)^2}u^{-q-3}u_{ij}u_i u_j v^\alpha \\ &-\frac{(n-1)\lambda}{(n-4)^2}u^{-q-4}|\nabla u|^4 v^\alpha - \lambda\alpha u^{-q-1}A_{ij}u_i v_j v^{\alpha-1} + \lambda\left(q-\frac{2}{n-4}\right)u^{-q-2}A_{ij}u_i u_j v^\alpha \\ &-\lambda u^{-q-1}J_i u_i v^\alpha - \frac{(n-4)\lambda}{4}|A|^2 u^{-q}v^\alpha + \frac{n-4}{2}\chi u^{-q-1}v^\alpha.\end{aligned}\tag{3.20}$$

Here we write $\Phi\sim\Psi$ to mean $\int_M \Phi d\mu = \int_M \Psi d\mu$. In view of (3.2) and (3.16) we see

$$\begin{aligned}&\alpha\int_M v^{\alpha-1}|\nabla v|^2 d\mu + \left[\frac{(\alpha-1)(q+1)}{\alpha+1} - \frac{\lambda}{n-4}\right]\int_M u^q v^{\alpha+2}d\mu \\ \le\ & \left[(q+1)\left(q+\frac{2}{\alpha+1}\right) - \frac{2\lambda}{(n-4)^2}\right]\int_M u^{-2}|\nabla u|^2 v^{\alpha+1}d\mu - \frac{\lambda}{n-4}\int_M u^{-q-2}\left|D^2u\right|^2 v^\alpha d\mu \\ &+\frac{2(n-2)\lambda}{(n-4)^2}\int_M u^{-q-3}u_{ij}u_i u_j v^\alpha d\mu - \frac{(n-1)\lambda}{(n-4)^2}\int_M u^{-q-4}|\nabla u|^4 v^\alpha d\mu \\ &+c\alpha\int_M v^{\alpha-\frac12}|\nabla v|\,d\mu + c\int_M v^{\alpha+1}d\mu + c\int_M v^\alpha d\mu.\end{aligned}\tag{3.21}$$

To continue we split D^2u into its trace component $\frac{\Delta u}{n}g$ and trace-free component Θ,

$$D^2 u = \Theta + \frac{\Delta u}{n}g.\tag{3.22}$$

Consequently by (3.17),

$$\left|D^2u\right|^2 = |\Theta|^2+\frac{1}{n}u^{2q+2}v^2-\frac{n-4}{n}Ju^{q+2}v+\frac{(n-4)^2}{4n}J^2u^2, \tag{3.23}$$

$$u_{ij}u_iu_j = -\frac{1}{n}u^{q+1}|\nabla u|^2 v+\frac{n-4}{2n}Ju|\nabla u|^2+\Theta_{ij}u_iu_j. \tag{3.24}$$

Plug these equalities into (3.21), we get

$$\begin{aligned} &\alpha\int_M v^{\alpha-1}|\nabla v|^2 d\mu+\left[\frac{(\alpha-1)(q+1)}{\alpha+1}-\frac{(n-1)\lambda}{n(n-4)}\right]\int_M u^q v^{\alpha+2}d\mu \qquad (3.25)\\ \leq\ & \left[(q+1)\left(q+\frac{2}{\alpha+1}\right)-\frac{4(n-1)\lambda}{n(n-4)^2}\right]\int_M u^{-2}|\nabla u|^2 v^{\alpha+1}d\mu\\ &-\frac{\lambda}{n-4}\int_M u^{-q-2}v^\alpha\left(|\Theta|^2-\frac{2(n-2)}{n-4}u^{-1}\Theta_{ij}u_iu_j+\frac{n-1}{n-4}u^{-2}|\nabla u|^4\right)d\mu\\ &+\frac{(n-2)\lambda}{n(n-4)}\int_M Ju^{-q-2}|\nabla u|^2 v^\alpha d\mu-\frac{(n-4)\lambda}{4n}\int_M J^2u^{-q}v^\alpha d\mu\\ &+c\alpha\int_M v^{\alpha-\frac{1}{2}}|\nabla v|\,d\mu+c\int_M v^{\alpha+1}d\mu+c\int_M v^\alpha d\mu. \end{aligned}$$

By (3.16) we have

$$\frac{(n-2)\lambda}{n(n-4)}\int_M Ju^{-q-2}|\nabla u|^2 v^\alpha d\mu\leq c\int_M v^{\alpha+1}d\mu. \tag{3.26}$$

This and

$$c\alpha\int_M v^{\alpha-\frac{1}{2}}|\nabla v|\,d\mu\leq\frac{\alpha}{2}\int_M v^{\alpha-1}|\nabla v|^2 d\mu+\frac{c^2\alpha}{2}\int_M v^\alpha \tag{3.27}$$

together with (3.25) implies for $\alpha\geq 1$,

$$\begin{aligned} &\frac{\alpha}{2}\int_M v^{\alpha-1}|\nabla v|^2 d\mu+\left[\frac{(\alpha-1)(q+1)}{\alpha+1}-\frac{(n-1)\lambda}{n(n-4)}\right]\int_M u^q v^{\alpha+2}d\mu \qquad (3.28)\\ \leq\ & \left[(q+1)\left(q+\frac{2}{\alpha+1}\right)-\frac{4(n-1)\lambda}{n(n-4)^2}\right]\int_M u^{-2}|\nabla u|^2 v^{\alpha+1}d\mu\\ &-\frac{\lambda}{n-4}\int_M u^{-q-2}v^\alpha\left(|\Theta|^2-\frac{2(n-2)}{n-4}u^{-1}\Theta_{ij}u_iu_j+\frac{n-1}{n-4}u^{-2}|\nabla u|^4\right)d\mu\\ &+c\int_M v^{\alpha+1}d\mu+c\alpha\int_M v^\alpha d\mu. \end{aligned}$$

Because Θ is trace-free,

$$|\Theta_{ij}u_iu_j|\leq\sqrt{\frac{n-1}{n}}|\Theta||\nabla u|^2, \tag{3.29}$$

hence

$$\begin{aligned}
& |\Theta|^2-\frac{2(n-2)}{n-4}u^{-1}\Theta_{ij}u_iu_j+\frac{n-1}{n-4}u^{-2}|\nabla u|^4 \\
\geq\ & |\Theta|^2-\frac{2(n-2)}{n-4}\sqrt{\frac{n-1}{n}}u^{-1}|\Theta||\nabla u|^2+\frac{n-1}{n-4}u^{-2}|\nabla u|^4 \\
=\ & \left(|\Theta|-\frac{n-2}{n-4}\sqrt{\frac{n-1}{n}}u^{-1}|\nabla u|^2\right)^2-\frac{4(n-1)}{n(n-4)^2}u^{-2}|\nabla u|^4 \\
\geq\ & -\frac{4(n-1)}{n(n-4)^2}u^{-2}|\nabla u|^4.
\end{aligned} \tag{3.30}$$

Plug this inequality into (3.28) we get

$$\begin{aligned}
& \frac{\alpha}{2}\int_M v^{\alpha-1}|\nabla v|^2 d\mu \\
& +\left[\frac{(\alpha-1)(q+1)}{\alpha+1}-\frac{(n-1)\lambda}{n(n-4)}\right]\int_M u^q v^{\alpha+2}d\mu \\
\leq\ & \left[(q+1)\left(q+\frac{2}{\alpha+1}\right)-\frac{4(n-1)\lambda}{n(n-4)^2}\right]\int_M u^{-2}|\nabla u|^2 v^{\alpha+1}d\mu \\
& +\frac{4(n-1)\lambda}{n(n-4)^3}\int_M u^{-q-4}|\nabla u|^4 v^\alpha d\mu+c\int_M v^{\alpha+1}d\mu+c\alpha\int_M v^\alpha d\mu.
\end{aligned} \tag{3.31}$$

By (3.16) we have

$$\frac{4(n-1)\lambda}{n(n-4)^3}\int_M u^{-q-4}|\nabla u|^4 v^\alpha d\mu\leq\frac{2(n-1)\lambda}{n(n-4)^2}\int_M u^{-2}|\nabla u|^2 v^{\alpha+1}d\mu, \tag{3.32}$$

hence

$$\begin{aligned}
& \frac{\alpha}{2}\int_M v^{\alpha-1}|\nabla v|^2 d\mu \\
& +\left[\frac{(\alpha-1)(q+1)}{\alpha+1}-\frac{(n-1)\lambda}{n(n-4)}\right]\int_M u^q v^{\alpha+2}d\mu \\
\leq\ & \left[(q+1)\left(q+\frac{2}{\alpha+1}\right)-\frac{2(n-1)\lambda}{n(n-4)^2}\right]\int_M u^{-2}|\nabla u|^2 v^{\alpha+1}d\mu \\
& +c\int_M v^{\alpha+1}d\mu+c\alpha\int_M v^\alpha d\mu.
\end{aligned} \tag{3.33}$$

By (3.16) again,

$$\int_M u^{-2}|\nabla u|^2 v^{\alpha+1}d\mu\leq\frac{n-4}{2}\int_M u^q v^{\alpha+2}d\mu, \tag{3.34}$$

we get

$$\begin{aligned}
&\frac{\alpha}{2}\int_M v^{\alpha-1}\left|\nabla v\right|^2 d\mu \\
&+\left[\frac{(\alpha-1)(q+1)}{\alpha+1}-\frac{(n-1)\lambda}{n(n-4)}\right]\int_M u^q v^{\alpha+2} d\mu \\
\leq\ & \max\left\{\frac{n-4}{2}(q+1)\left(q+\frac{2}{\alpha+1}\right)-\frac{(n-1)\lambda}{n(n-4)},0\right\}\int_M u^q v^{\alpha+2} d\mu \\
&+c\int_M v^{\alpha+1} d\mu+c\alpha\int_M v^{\alpha} d\mu.
\end{aligned} \tag{3.35}$$

In another word

$$\begin{aligned}
&\frac{\alpha}{2}\int_M v^{\alpha-1}\left|\nabla v\right|^2 d\mu \\
&+\min\left\{\frac{(\alpha-1)(q+1)}{\alpha+1}-\frac{(n-1)\lambda}{n(n-4)},(q+1)\left(-\frac{n-4}{2}q+\frac{\alpha-n+3}{\alpha+1}\right)\right\}\int_M u^q v^{\alpha+2} d\mu \\
\leq\ & c\int_M v^{\alpha+1} d\mu+c\alpha\int_M v^{\alpha} d\mu.
\end{aligned} \tag{3.36}$$

Because $n\geq 6$, we see

$$\frac{n-2}{n-4}>\frac{4(n-1)}{n(n-4)}. \tag{3.37}$$

Fix a $q\geq 0$ such that

$$\frac{n-2}{n-4}>q+1>\frac{4(n-1)}{n(n-4)}, \tag{3.38}$$

then for α large enough,

$$\alpha\int_M v^{\alpha-1}\left|\nabla v\right|^2 d\mu+\int_M u^q v^{\alpha+2} d\mu\leq c\int_M v^{\alpha+1} d\mu+c\alpha\int_M v^{\alpha} d\mu. \tag{3.39}$$

By (3.2) and (3.39) we get

$$\left\|v\right\|_{L^{\alpha+2}}^{\alpha+2}\leq c(\alpha)\left(\left\|v\right\|_{L^{\alpha+2}}^{\alpha+1}+\left\|v\right\|_{L^{\alpha+2}}^{\alpha}\right). \tag{3.40}$$

Hence

$$\left\|v\right\|_{L^{\alpha+2}}\leq c(\alpha) \tag{3.41}$$

for α large enough. To continue, we observe that for α large enough,

$$\int_M\left|\nabla v^{\frac{\alpha+1}{2}}\right|^2 d\mu\leq c\alpha\int_M v^{\alpha+1} d\mu+c\alpha^2\int_M v^{\alpha} d\mu. \tag{3.42}$$

Hence

$$\begin{aligned}
\left\|v^{\frac{\alpha+1}{2}}\right\|_{L^{\frac{2n}{n-2}}}^2 &\leq c\left(\int_M\left|\nabla v^{\frac{\alpha+1}{2}}\right|^2 d\mu+\int_M v^{\alpha+1} d\mu\right) \\
&\leq c\alpha\left\|v\right\|_{L^{\alpha+1}}^{\alpha+1}+c\alpha^2\left\|v\right\|_{L^{\alpha+1}}^{\alpha}.
\end{aligned} \tag{3.43}$$

Replacing $\alpha+1$ by α we see

$$\left\|v\right\|_{L^{\kappa\alpha}}^{\alpha}\leq c\alpha\left\|v\right\|_{L^{\alpha}}^{\alpha}+c\alpha^2\left\|v\right\|_{L^{\alpha}}^{\alpha-1}. \tag{3.44}$$

Here

$$\kappa=\frac{n}{n-2}. \tag{3.45}$$

Let

$$b_\alpha = \max\left\{\|v\|_{L^\alpha}, 1\right\}, \tag{3.46}$$

then for α large enough, we have

$$b_{\kappa\alpha}^\alpha \le c\alpha^2 b_\alpha^\alpha. \tag{3.47}$$

It follows from iteration that for a fixed α_0 large,

$$b_{\kappa^k\alpha_0} \le c b_{\alpha_0} \le c. \tag{3.48}$$

Hence

$$\|v\|_{L^{\kappa^k\alpha_0}} \le c. \tag{3.49}$$

Letting $k \to \infty$ we see

$$\|v\|_{L^\infty} \le c. \tag{3.50}$$

Now we go back to the equation of u,

$$-\Delta u + \frac{n-4}{2} J u = v u^{q+1}. \tag{3.51}$$

Since $\|u\|_{L^{\frac{2n}{n-4}}} \le c$ and $q < \frac{4}{n-4}$, standard bootstrap method tells us

$$\|u\|_{L^\infty} \le c. \tag{3.52}$$

Elliptic estimate tells us

$$\|u\|_{W^{2,p}} \le c(p) \tag{3.53}$$

for any $1 < p < \infty$. This together with (2.8) and (3.2) imply

$$\|u\|_{C^k} \le c(k) \tag{3.54}$$

for all $k \in \mathbb{N}$.

Proposition 3.1. *Under the assumption of Theorem 1.1, the set Σ is closed.*

Proof. Assume $\lambda_i \in \Sigma$, $\lambda_i \to \lambda_\infty$ as $i \to \infty$, u_i is a solution to (2.4) for $\lambda = \lambda_i$ satisfying $J_i > 0$, then

$$\|u_i\|_{C^k} \le c(k),$$

for all $k \in \mathbb{N}$ and

$$u_i \ge c > 0.$$

After passing to a subsequence we have $u_i \to u_\infty$ in C^∞ and $u_\infty \ge c > 0$. Denote

$$g_i = u_i^{\frac{4}{n-4}} g, \quad g_\infty = u_\infty^{\frac{4}{n-4}} g. \tag{3.55}$$

Then

$$Q_i - \lambda_i \sigma_2(A_i) = \chi u_i^{-\frac{n+4}{n-4}}, \quad J_i > 0.$$

Let $i \to \infty$, we get

$$Q_\infty - \lambda_\infty \sigma_2(A_\infty) = \chi u_\infty^{-\frac{n+4}{n-4}}, \quad J_\infty \ge 0. \tag{3.56}$$

In another way,

$$-\Delta_\infty J_\infty - \frac{4-\lambda_\infty}{2}|A_\infty|_\infty^2 + \frac{n-\lambda_\infty}{2} J_\infty^2 = \chi u_\infty^{-\frac{n+4}{n-4}} > 0. \tag{3.57}$$

Hence

$$-\Delta_\infty J_\infty + \frac{n-\lambda_\infty}{2} J_\infty^2 > 0. \tag{3.58}$$

Since $J_\infty \ge 0$, it follows from strong maximum principle that either $J_\infty > 0$ or $J_\infty \equiv 0$. The latter case contradicts with (3.58). Hence $J_\infty > 0$ and $\lambda_\infty \in \Sigma$. It follows that Σ is closed. ∎

4. Appendix

Here we give an elementary proof of Proposition 2.1. Assume $Y(M,g) > 0$, we can assume $R > 0$. For $1 < p < \frac{n+2}{n-2}$, based on the compact embedding $H^1(M) \subset L^{p+1}(M)$ we know there exists a positive smooth function u satisfying

$$Lu = u^p \tag{4.1}$$

(see [LP]). Here L is the conformal Laplacian operator (1.2). Let

$$\widetilde{g} = u^{\frac{4}{n-2}} g, \tag{4.2}$$

then

$$\widetilde{R} = u^{-\frac{n+2}{n-2}} Lu = u^{p-\frac{n+2}{n-2}} > 0, \tag{4.3}$$

hence $\widetilde{J} > 0$. Note that

$$-\widetilde{\Delta}\widetilde{J} + \frac{n-4}{2}\widetilde{J}^2 = \frac{1}{2(n-1)}\left(-\widetilde{\Delta}\widetilde{R} + \frac{n-4}{4(n-1)}\widetilde{R}^2\right). \tag{4.4}$$

Since

$$\widetilde{\Delta}\varphi = u^{-\frac{4}{n-2}}\Delta\varphi + 2u^{-\frac{n+2}{n-2}} g(\nabla u, \nabla\varphi), \tag{4.5}$$

using (4.1) and (4.3) we have

$$\begin{aligned} & -\widetilde{\Delta}\widetilde{R} + \frac{n-4}{4(n-1)}\widetilde{R}^2 \qquad (4.6)\\ = & -\left(u^{-\frac{4}{n-2}}\Delta\widetilde{R} + 2u^{-\frac{n+2}{n-2}} g\left(\nabla u, \nabla\widetilde{R}\right)\right) + \frac{n-4}{4(n-1)}\widetilde{R}^2 \\ = & \frac{n-2}{4(n-1)}\left(p - \frac{6}{n-2}\right) u^{2p-\frac{2(n+2)}{n-2}} + \frac{n-2}{4(n-1)}\left(\frac{n+2}{n-2} - p\right) R u^{p-\frac{n+6}{n-2}} \\ & + \left(\frac{n+2}{n-2} - p\right)\left(p - \frac{4}{n-2}\right) u^{p-\frac{3n+2}{n-2}} |\nabla u|^2 . \end{aligned}$$

We choose p such that

$$\max\left\{1, \frac{6}{n-2}\right\} < p < \frac{n+2}{n-2}, \tag{4.7}$$

this is possible since $n \geq 5$. Then it follows from (4.4) and (4.6) that

$$-\widetilde{\Delta}\widetilde{J} + \frac{n-4}{2}\widetilde{J}^2 > 0. \tag{4.8}$$

References

[B] T. Branson. *Differential operators canonically associated to a conformal structure*. Math. Scand. **57** (1985), no. 2, 293–345.

[CGY] S.-Y. A. Chang, M. J. Gursky and P. C. Yang. *An equation of Monge-Ampere type in conformal geometry, and four-manifolds of positive Ricci curvature*. Ann. of Math. (2) **155** (2002), 709–787.

[GT] D. Gilbarg and N. S. Trudinger. Elliptic partial differential equations of second order. 2nd edition, 3rd printing, Berlin, Springer-Verlag, 1998.

[GM] M. J. Gursky and A. Malchiodi. *A strong maximum principle for the Paneitz operator and a nonlocal flow for the* Q *curvature*. J Eur Math Soc, to appear.

[HY1] F. B. Hang and P. Yang. *Sign of Green's function of Paneitz operators and the* Q *curvature*. International Mathematics Research Notices 2014; doi: 10.1093/imrn/rnu247.

[HY2] F. B. Hang and P. Yang. Q curvature on a class of manifolds with dimension at least 5. Preprint (2014).

[HeR] E. Hebey and F. Robert. *Compactness and global estimates for the geometric Paneitz equation in high dimensions*. Electron Res Ann Amer Math Soc. **10** (2004), 135–141.

[HuR] E. Humbert and S. Raulot. *Positive mass theorem for the Paneitz-Branson operator.* Calculus of Variations and PDE. **36** (2009), 525–531.

[LP] J. M. Lee and T. H. Parker. *The Yamabe problem.* Bull AMS. **17** (1987), no. 1, 37–91.

[P] S. Paneitz. A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds. Preprint (1983).

[R] F. Robert. Fourth order equations with critical growth in Riemannian geometry. Unpublished notes. Available at http://www.iecn.u-nancy.fr/~frobert/LectRobertFourth.pdf.

[XY] X. W. Xu and P. Yang. *Positivity of Paneitz operators.* Discrete Contin. Dynam. Systems **7** (2001), no. 2, 329–342.

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